

# SOME IDENTITIES OF DEGENERATE ORDERED BELL POLYNOMIALS AND NUMBERS ARISING FROM UMBRAL CALCULUS

TAEKYUN KIM AND DAE SAN KIM

ABSTRACT. In this paper, we study degenerate ordered Bell polynomials with the viewpoint of Carlitz's degenerate Bernoulli and Euler polynomials and derive by using umbral calculus some properties and new identities for the degenerate ordered Bell polynomials associated with special polynomials.

## 1. Introduction

As is well known, the ordinary Euler polynomials are defined by the generating function

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (\text{see } [6, 7, 8]). \quad (1.1)$$

When  $x = 0$ ,  $E_n = E_n(0)$ , ( $n \geq 0$ ), are called the Euler numbers. The Bernoulli polynomials are also given by the generating function as follows:

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (\text{see } [5, 6, 8]). \quad (1.2)$$

When  $x = 0$ ,  $B_n = B_n(0)$ , are called the Bernoulli numbers. For  $\lambda \in \mathbb{R}$ , L. Carlitz considered the degenerate Euler and Bernoulli polynomials which are given by the generating function

$$\frac{2}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_n(x|\lambda) \frac{t^n}{n!}, \quad (1.3)$$

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and

$$\frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(x|\lambda) \frac{t^n}{n!}, \quad (\text{see [2]}). \quad (1.4)$$

Note that  $\lim_{\lambda \rightarrow 0} \mathcal{E}_n(x|\lambda) = E_n(x)$  and  $\lim_{\lambda \rightarrow 0} \beta_n(x|\lambda) = B_n(x)$ , ( $n \geq 0$ ), (see [2,7]) The falling factorial sequences are defined by

$$(x)_0 = 1, \quad (x)_n = x(x-1) \cdots (x-(n-1)), \quad (n \geq 1). \quad (1.5)$$

The Stirling numbers of the first kind are defined as

$$(x)_n = \sum_{l=0}^n S_1(n, l) x^l, \quad (n \geq 0), \quad (\text{see [10]}). \quad (1.6)$$

The Stirling numbers of the second kind are also defined by

$$x^n = \sum_{l=0}^n S_2(n, l) (x)_l, \quad (n \geq 0), \quad (\text{see [6, 10]}). \quad (1.7)$$

It is well known that the ordered Bell polynomials are defined by the generating function

$$\frac{1}{2 - e^t} e^{xt} = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!}, \quad (\text{see [3]}). \quad (1.8)$$

When  $x = 0$ ,  $b_n = b_n(0)$  are called the ordered Bell numbers. From (1.8), we note that

$$\begin{aligned} \frac{1}{2 - e^t} &= \frac{1}{1 - (e^t - 1)} = \sum_{m=0}^{\infty} (e^t - 1)^m \\ &= \sum_{m=0}^{\infty} m! \sum_{n=m}^{\infty} S_2(n, m) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{m=0}^n m! S_2(n, m) \right) \frac{t^n}{n!}. \end{aligned} \quad (1.9)$$

Thus, by (1.9), we get

$$b_n = \sum_{m=0}^n m! S_2(n, m), \quad (n \geq 0). \quad (1.10)$$

For  $r \in \mathbb{N}$ , the higher-order ordered Bell polynomials are given by the generating function

$$\left( \frac{1}{2 - e^t} \right)^r e^{xt} = \sum_{n=0}^{\infty} b_n^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [3]}). \quad (1.11)$$

When  $x = 0$ ,  $b_n^{(r)} = b_n^{(r)}(0)$ , ( $n \geq 0$ ), are called the higher-order ordered Bell numbers. By (1.11), we easily get

$$\begin{aligned}
\sum_{n=0}^{\infty} b_n^{(r)} \frac{t^n}{n!} &= (2 - e^t)^{-r} = (1 - (e^t - 1))^{-r} \\
&= \sum_{m=0}^{\infty} \binom{m+r-1}{m} (e^t - 1)^m \\
&= \sum_{m=0}^{\infty} \binom{m+r-1}{m} m! \sum_{n=m}^{\infty} S_2(n, m) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left( \sum_{m=0}^n \binom{m+r-1}{m} m! S_2(n, m) \right) \frac{t^n}{n!}
\end{aligned} \tag{1.12}$$

From (1.12), we note that

$$b_n^{(r)} = \sum_{m=0}^n \binom{m+r-1}{m} m! S_2(n, m), \quad (n \geq 0). \tag{1.13}$$

In view of (1.3) and (1.4), we consider the degenerate ordered Bell polynomials given by the generating function

$$\frac{1}{2 - (1 + \lambda t)^{\frac{1}{\lambda}}} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} b_{n,\lambda}(x) \frac{t^n}{n!}. \tag{1.14}$$

By (1.14), we easily get

$$\lim_{\lambda \rightarrow 0} b_{n,\lambda}(x) = b_n(x), \quad (n \geq 0). \tag{1.15}$$

When  $x = 0$ ,  $b_{n,\lambda} = b_{n,\lambda}(0)$  are called the degenerate ordered Bell numbers. Replacing  $t$  by  $\frac{1}{\lambda}(e^{\lambda t} - 1)$ , we get

$$\begin{aligned}
\sum_{m=0}^{\infty} b_{m,\lambda}(x) \frac{1}{m!} \left( \frac{1}{\lambda} (e^{\lambda t} - 1) \right)^m &= \frac{1}{2 - e^t} e^{xt} \\
&= \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!}.
\end{aligned} \tag{1.16}$$

Now, we observe that

$$\begin{aligned}
& \sum_{m=0}^{\infty} b_{m,\lambda}(x) \frac{1}{m!} \lambda^{-m} (e^{\lambda t} - 1)^m \\
&= \sum_{m=0}^{\infty} b_{m,\lambda}(x) \lambda^{-m} \sum_{n=m}^{\infty} S_2(n, m) \lambda^n \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left( \sum_{m=0}^n \lambda^{n-m} b_{m,\lambda}(x) S_2(n, m) \right) \frac{t^n}{n!}.
\end{aligned} \tag{1.17}$$

From (1.16) and (1.17), we have

$$b_n(x) = \sum_{m=0}^n \lambda^{n-m} b_{m,\lambda}(x) S_2(n, m). \tag{1.18}$$

Here we study degenerate ordered Bell polynomials with the viewpoint of Carlitz's degenerate Bernoulli and Euler polynomials and derive by using umbral calculus some properties and new identities for the degenerate ordered Bell polynomials associated with special polynomials.

## 2. Quick review of umbral calculus

Let  $\mathbb{C}$  be the complex number field and let  $\mathcal{F}$  be the set of all formal power series with coefficients in  $\mathbb{C}$  in the variable  $t$ :

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\}. \tag{2.1}$$

We denote the algebra  $\mathbb{C}[x]$  of polynomials in  $x$  over the field  $\mathbb{C}$  by  $\mathbb{P}$  ( $\mathbb{P} = \mathbb{C}[x]$ ). Let  $\mathbb{P}^*$  be the vector space of all linear functionals on  $\mathbb{P}$ , and let  $\langle L | p(x) \rangle$  denote the action of the linear functional  $L$  on  $p(x)$ , which satisfies  $\langle L + M | p(x) \rangle = \langle L | p(x) \rangle + \langle M | p(x) \rangle$  and  $\langle cL | p(x) \rangle = c \langle L | p(x) \rangle$ , where  $c$  is a complex number. The linear functional  $\langle f(t) | \cdot \rangle$  on  $\mathbb{P}$  is defined by  $\langle f(t) | x^n \rangle = a_n$  ( $n \geq 0$ ), where  $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$ . Thus we note that  $\langle t^k | x^n \rangle = n! \delta_{n,k}$  ( $n, k \geq 0$ ), where  $\delta_{n,k}$  is the Kronecker symbol (see [1,4,10]). Let  $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L | x^k \rangle}{k!} t^k$ . Then we have  $\langle f_L(t) | x^n \rangle = \langle L | x^n \rangle$  ( $n \geq 0$ ). So, the map  $L \mapsto f_L(t)$  is a vector space isomorphism from  $\mathbb{P}^*$  onto  $\mathcal{F}$ . Henceforth,  $\mathcal{F}$  denotes both the algebra of formal power series in  $t$  and the vector space of all linear functionals on  $\mathbb{P}$ , and so an element  $f(t)$  of  $\mathcal{F}$  will be thought of as both a formal power series and a linear functional. We call  $\mathcal{F}$  the umbral algebra. The umbral calculus is the study of the umbral algebra. Let  $f(t) (\neq 0) \in \mathcal{F}$ . Then order of  $f(t)$  is the smallest positive integer  $k$  for which

the coefficient of  $t^k$  does not vanish. The order of  $f(t)$  is denoted by  $o(f(t))$ , (see [7,8,10]). For  $f(t) \in \mathcal{F}$  and  $p(x) \in \mathbb{P}$ , we have

$$f(t) = \sum_{k=0}^{\infty} \langle f(t)|x^k \rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \langle t^k|p(x) \rangle \frac{t^k}{k!}. \quad (2.2)$$

Thus, by (2.2), we easily get

$$p^{(k)}(0) = \langle t^k|p(x) \rangle = \langle 1|p^{(k)}(x) \rangle, \quad (\text{see [9, 10]}), \quad (2.3)$$

where  $p^{(k)}(x) = \left(\frac{d}{dx}\right)^k p(x)$ . From (2.3), we note that

$$t^k p(x) = p^{(k)}(x), \quad e^{yt} p(x) = p(x+y), \quad \text{and} \quad \langle e^{yt}|p(x) \rangle = p(y), \quad (\text{see [10, 11, 12]}). \quad (2.4)$$

For  $f(t), g(t) \in \mathcal{F}$  with  $o(f(t)) = 1$ ,  $o(g(t)) = 0$ , there exists a unique sequence  $S_n(x)$  of polynomials with  $\deg S_n(x) = n$  such that

$$\langle g(t)f(t)^k|S_n(x) \rangle = n! \delta_{n,k}, \quad (n, k \geq 0), \quad (\text{see [6, 7, 10]}). \quad (2.5)$$

The sequence  $S_n(x)$  is called the Sheffer sequence for  $(g(t), f(t))$ , and we write  $S_n(x) \sim (g(t), f(t))$ . It is well known that

$$S_n(x) \sim (g(t), f(t)) \iff \frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} S_n(x) \frac{t^n}{n!},$$

where  $\bar{f}(t)$  is the compositional inverse of  $f(t)$  such that  $f(\bar{f}(t)) = \bar{f}(f(t)) = t$ . Let  $S_n(x) \sim (g(t), f(t))$ . Then we have

$$f(t)s_n(x) = nS_{n-1}(x), \quad (n \geq 1), \quad S_n(x) = \sum_{j=0}^n \frac{\langle g(\bar{f}(t))^{-1} \bar{f}(t)^j | x^n \rangle}{j!} x^j, \quad (2.6)$$

$$S_n(x+y) = \sum_{j=0}^n \binom{n}{j} S_j(x) P_{n-j}(y), \quad \text{where } P_n(y) = g(t)S_n(y), \quad (2.7)$$

$$\langle f(t)|xp(x) \rangle = \langle \delta_t f(t)|p(x) \rangle, \quad \text{where } \delta_t f(t) = \frac{d}{dt} f(t), \quad (2.8)$$

and

$$S_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right) \frac{1}{f'(t)} S_n(x), \quad (n \geq 0), \quad (\text{see [10]}). \quad (2.9)$$

For  $p_n(x) \sim (1, f(t))$ ,  $q_n(x) \sim (1, g(t))$ , we have

$$q_n(x) = x \left(\frac{f(t)}{g(t)}\right)^n x^{-1} p_n(x), \quad (n \geq 1), \quad (\text{see [10, 12]}). \quad (2.10)$$

Let us consider the following two Sheffer sequences:

$$S_n(x) \sim (g(t), f(t)), \quad r_n(x) \sim (h(t), l(t)). \quad (2.11)$$

Then we have

$$S_n(x) = \sum_{m=0}^n C_{n,m} r_m(x), \quad (n \geq 0), \quad (2.12)$$

where

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} \left( l(\bar{f}(t)) \right)^m |x^n \right\rangle, \quad (\text{see [10]}). \quad (2.13)$$

Finally, we recall that

$$\frac{d}{dx} S_n(x) = \sum_{k=0}^{n-1} \binom{n}{k} \langle \bar{f}(t) | x^{n-k} \rangle S_k(x). \quad (2.14)$$

### 3. Degenerate ordered Bell numbers and polynomials associated with umbral calculus

From (1.14), we note that the degenerate ordered Bell polynomials are defined by the generating function

$$\frac{1}{2 - (1 + \lambda t)^{\frac{1}{\lambda}}} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} b_{n,\lambda}(x) \frac{t^n}{n!}.$$

When  $x = 0$ ,  $b_{n,\lambda} = b_{n,\lambda}(0)$  are called the ordered Bell numbers. By (1.14), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{n,\lambda} \frac{t^n}{n!} &= \frac{1}{2 - (1 + \lambda t)^{\frac{1}{\lambda}}} = \frac{1}{1 - ((1 + \lambda t)^{\frac{1}{\lambda}} - 1)} \\ &= \sum_{m=0}^{\infty} \left( (1 + \lambda t)^{\frac{1}{\lambda}} - 1 \right)^m = \sum_{m=0}^{\infty} \left( e^{\frac{1}{\lambda} \log(1 + \lambda t)} - 1 \right)^m \\ &= \sum_{m=0}^{\infty} m! \sum_{k=m}^{\infty} S_2(k, m) \lambda^{-k} \frac{1}{k!} (\log(1 + \lambda t))^k \\ &= \sum_{k=0}^{\infty} \left( \sum_{m=0}^k m! S_2(k, m) \lambda^{-k} \right) \sum_{n=k}^{\infty} S_1(n, k) \frac{\lambda^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \sum_{m=0}^k m! S_2(k, m) S_1(n, k) \lambda^{n-k} \right) \frac{t^n}{n!}. \end{aligned} \quad (3.1)$$

From (3.1), we note that

$$b_{n,\lambda} = \sum_{k=0}^n \sum_{m=0}^k m! S_2(k, m) S_1(n, k) \lambda^{n-k}, \quad (n \geq 0). \quad (3.2)$$

By (1.14) and (2.5), we get

$$b_{n,\lambda}(x) \sim (2 - e^t, \frac{1}{\lambda}(e^{\lambda t} - 1)), \quad (n \geq 0). \quad (3.3)$$

That is

$$\frac{1}{2 - e^{\frac{1}{\lambda} \log(1+\lambda t)}} e^{x \frac{1}{\lambda} \log(1+\lambda t)} = \sum_{n=0}^{\infty} b_{n,\lambda}(x) \frac{t^n}{n!}. \quad (3.4)$$

Let  $f(t)$  be the linear functional such that

$$\langle f(t) | p(x) \rangle = \int_0^y p(u) du. \quad \text{for all } p(x). \quad (3.5)$$

Then, by (2.2) and (3.5), we get

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} \frac{\langle f(t) | x^k \rangle}{k!} t^k = \sum_{k=0}^{\infty} \frac{y^{k+1}}{(k+1)!} t^k \\ &= \frac{1}{t} (e^{yt} - 1). \end{aligned} \quad (3.6)$$

From (3.5), we have

$$\langle \frac{1}{t} (e^{yt} - 1) | p(x) \rangle = \int_0^y p(u) du, \quad (3.7)$$

and

$$\frac{1}{t} (e^{yt} - 1) p(x) = \int_x^{x+y} p(u) du. \quad (3.8)$$

By (3.1), (3.4), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} b_{n,\lambda}(x) \frac{t^n}{n!} &= \frac{1}{2 - (1 + \lambda t)^{\frac{1}{\lambda}}} e^{\frac{x}{\lambda} \log(1 + \lambda t)} \\
&= \left( \sum_{l=0}^{\infty} b_{l,\lambda} \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} \left( \frac{x}{\lambda} \right)^m \frac{1}{m!} (\log(1 + \lambda t))^m \right) \\
&= \left( \sum_{l=0}^{\infty} b_{l,\lambda} \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} \left( \frac{x}{\lambda} \right)^m \sum_{k=m}^{\infty} S_1(k, m) \frac{\lambda^k t^k}{k!} \right) \\
&= \left( \sum_{l=0}^{\infty} b_{l,\lambda} \frac{t^l}{l!} \right) \left( \sum_{k=0}^{\infty} \left( \sum_{m=0}^k \lambda^{k-m} S_1(k, m) x^m \right) \frac{t^k}{k!} \right) \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^n \sum_{m=0}^k \lambda^{k-m} S_1(k, m) \binom{n}{k} x^m b_{n-k,\lambda} \right\} \frac{t^n}{n!}.
\end{aligned} \tag{3.9}$$

Comparing the coefficients on both sides of (3.9), we have

$$b_{n,\lambda}(x) = \sum_{k=0}^n \sum_{m=0}^k \binom{n}{k} \lambda^{k-m} S_1(k, m) b_{n-k,\lambda} x^m. \tag{3.10}$$

From (3.8), we note that

$$\begin{aligned}
\frac{e^t - 1}{t} b_{n,\lambda}(x) &= \int_x^{x+1} b_{n,\lambda}(u) du \\
&= \sum_{k=0}^n \sum_{m=0}^k \binom{n}{k} \lambda^{k-m} S_1(k, m) b_{n-k,\lambda} \int_x^{x+1} u^m du \\
&= \sum_{k=0}^n \sum_{m=0}^k \binom{n}{k} \lambda^{k-m} S_1(k, m) b_{n-k,\lambda} \frac{1}{m+1} ((x+1)^{m+1} - x^{m+1}) \\
&= \sum_{k=0}^n \sum_{m=0}^k \sum_{l=0}^m \binom{m+1}{l} \binom{n}{k} \lambda^{k-m} S_1(k, m) b_{n-k,\lambda} \frac{1}{m+1} x^l.
\end{aligned} \tag{3.11}$$

As is well known,  $B_n(x) = \frac{t}{e^t - 1} x^n$ , ( $n \geq 0$ ). By (3.11), we get

$$b_{n,\lambda}(x) = \sum_{k=0}^n \sum_{m=0}^k \sum_{l=0}^m \lambda^{k-m} \binom{m+1}{l} \binom{n}{k} S_1(k, m) b_{n-k,\lambda} \frac{1}{m+1} B_l(x). \tag{3.12}$$



From (3.12), we note that

$$\begin{aligned}
 tb_{n,\lambda}(x) &= \sum_{k=1}^n \sum_{m=1}^k \sum_{l=1}^m \lambda^{k-m} \binom{m+1}{l} \binom{n}{k} \frac{l}{m+1} B_{l-1}(x) S_1(k, m) b_{n-k,\lambda} \\
 &= \sum_{k=1}^n \sum_{m=1}^k \sum_{l=1}^m \lambda^{k-m} \binom{m}{l-1} \binom{n}{k} B_{l-1}(x) S_1(k, m) b_{n-k,\lambda}.
 \end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
 \frac{e^{yt} - 1}{t} b_{n,\lambda}(x) &= \int_x^{x+y} b_{n,\lambda}(u) du \\
 &= \sum_{k=0}^n \sum_{m=0}^k \sum_{l=0}^m \lambda^{k-m} \binom{m+1}{l} \binom{n}{k} S_1(k, m) b_{n-k,\lambda} \left( \frac{B_{l+1}(x+y) - B_{l+1}(x)}{(m+1)(l+1)} \right) \\
 &= \sum_{k=0}^n \sum_{m=0}^k \sum_{l=0}^m \sum_{j=0}^l \binom{m+1}{l} \binom{n}{k} \binom{l+1}{j} \lambda^{k-m} S_1(k, m) b_{n-k,\lambda} y^{l+1-j} B_j(x) \frac{1}{(m+1)(l+1)} \\
 &= \sum_{k=0}^n \sum_{m=0}^k \sum_{l=0}^m \sum_{j=0}^l \frac{\binom{m+1}{l} \binom{n}{k} \binom{l+1}{j}}{(m+1)(l+1)} \lambda^{k-m} S_1(k, m) b_{n-k,\lambda} y^{l+1-j} B_j(x).
 \end{aligned} \tag{3.14}$$

By multiplying  $t$  on both sides of (3.14), we get

$$\begin{aligned}
 b_{n,\lambda}(x+y) - b_{n,\lambda}(x) &= (e^{yt} - 1) b_{n,\lambda}(x) \\
 &= \sum_{k=1}^n \sum_{m=1}^k \sum_{l=1}^m \sum_{j=1}^l \frac{\binom{m+1}{l} \binom{n}{k} \binom{l+1}{j} j}{(m+1)(l+1)} \lambda^{k-m} S_1(k, m) b_{n-k,\lambda} y^{l+1-j} B_{j-1}(x) \\
 &= \sum_{k=1}^n \sum_{m=1}^k \sum_{l=1}^m \sum_{j=1}^l \frac{\binom{m+1}{l} \binom{n}{k} \binom{l}{j-1}}{m+1} \lambda^{k-m} S_1(k, m) b_{n-k,\lambda} y^{l+1-j} B_{j-1}(x).
 \end{aligned} \tag{3.15}$$

Therefore, by (3.15), we obtain the following theorem.

**Theorem 3.1.** *For  $n \geq 1$ , we have*

$$\begin{aligned}
 b_{n,\lambda}(x+y) - b_{n,\lambda}(x) &= \sum_{k=1}^n \sum_{m=1}^k \sum_{l=1}^m \sum_{j=1}^l \frac{\binom{m+1}{l} \binom{n}{k} \binom{l}{j-1}}{m+1} \lambda^{k-m} S_1(k, m) b_{n-k,\lambda} y^{l+1-j} B_{j-1}(x).
 \end{aligned}$$

Now, we observe that

$$\begin{aligned}
& \sum_{n=0}^{\infty} (-b_{n,\lambda}(x+1) + 2b_{n,\lambda}(x)) \frac{t^n}{n!} = \frac{(1+\lambda t)^{\frac{x}{\lambda}} (2 - (1+\lambda t)^{\frac{1}{\lambda}})}{2 - (1+\lambda t)^{\frac{1}{\lambda}}} \\
& = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{k=0}^{\infty} \left(\frac{x}{\lambda}\right)^k \frac{1}{k!} (\log(1+\lambda t))^k \\
& = \sum_{k=0}^{\infty} \left(\frac{x}{\lambda}\right)^k \sum_{n=k}^{\infty} S_1(n, k) \lambda^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \lambda^{n-k} S_1(n, k) x^k \right) \frac{t^n}{n!}.
\end{aligned} \tag{3.16}$$

Comparing the coefficients on both sides of (3.16), we obtain the following theorem.

**Theorem 3.2.** *For  $n \geq 0$ , we have*

$$2b_{n,\lambda}(x) - b_{n,\lambda}(x+1) = \sum_{k=0}^n \lambda^{n-k} S_1(n, k) x^k.$$

From Theorem 1 and Theorem 2, we note that

$$\begin{aligned}
b_{n,\lambda}(x+1) - b_{n,\lambda}(x) &= b_{n,\lambda}(x) - \sum_{k=0}^n \lambda^{n-k} S_1(n, k) x^k \\
&= \sum_{k=1}^n \sum_{m=1}^k \sum_{l=1}^m \sum_{j=1}^l \frac{\binom{m+1}{l} \binom{n}{k} \binom{l}{j-1}}{m+1} \lambda^{k-m} S_1(k, m) b_{n-k,\lambda} B_{j-1}(x).
\end{aligned} \tag{3.17}$$

Therefore, by (3.17), we obtain the following corollary.

**Corollary 3.3.** *For  $n \geq 1$ , we have*

$$\begin{aligned}
b_{n,\lambda}(x) &= \sum_{k=1}^n \lambda^{n-k} S_1(n, k) x^k \\
&+ \sum_{k=1}^n \sum_{m=1}^k \sum_{l=1}^m \sum_{j=1}^l \frac{\binom{m+1}{l} \binom{n}{k} \binom{l}{j-1}}{m+1} \lambda^{k-m} S_1(k, m) b_{n-k,\lambda} B_{j-1}(x).
\end{aligned}$$

From  $\langle t^k | x^n \rangle = n! \delta_{n,k}$ , ( $n, k \geq 0$ ), we note that

$$\begin{aligned}
b_{n,\lambda}(y) &= \left\langle \sum_{l=0}^{\infty} b_{l,\lambda}(y) \frac{t^l}{l!} \middle| x^n \right\rangle = \left\langle \frac{1}{2 - (1 + \lambda t)^{\frac{1}{\lambda}}} (1 + \lambda t)^{\frac{y}{\lambda}} \middle| x^n \right\rangle \\
&= \left\langle \frac{1}{2 - (1 + \lambda t)^{\frac{1}{\lambda}}} \middle| (1 + \lambda t)^{\frac{y}{\lambda}} x^n \right\rangle \\
&= \sum_{m=0}^n \binom{n}{m} \sum_{k=0}^m y^k \lambda^{m-k} S_1(m, k) \left\langle \frac{1}{2 - (1 + \lambda t)^{\frac{1}{\lambda}}} \middle| x^{n-m} \right\rangle \quad (3.18) \\
&= \sum_{m=0}^n \binom{n}{m} \sum_{k=0}^m y^k \lambda^{m-k} S_1(m, k) \sum_{l=0}^{n-m} \frac{b_{l,\lambda}}{l!} \langle t^l | x^{n-m} \rangle \\
&= \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \lambda^{m-k} S_1(m, k) y^k b_{n-m,\lambda}.
\end{aligned}$$

Therefore, by (3.18), we obtain the following theorem.

**Theorem 3.4.** *For  $n \geq 0$ , we have*

$$b_{n,\lambda}(x) = \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \lambda^{m-k} S_1(m, k) b_{n-m,\lambda} x^k.$$

We easily see that

$$x^n \sim (1, t), \quad (2 - e^t) b_{n,\lambda}(x) \sim \left(1, \frac{1}{\lambda} (e^{\lambda t} - 1)\right). \quad (3.19)$$

Then, by (2.10) and (3.19), we get

$$\begin{aligned}
(2 - e^t) b_{n,\lambda}(x) &= x \left( \frac{\lambda t}{e^{\lambda t} - 1} \right)^n x^{-1} x^n \\
&= x \sum_{l=0}^{\infty} \lambda^l \frac{B_l^{(n)}}{l!} t^l x^{n-1} \\
&= x \sum_{l=0}^{n-1} \lambda^l \binom{n-1}{l} B_l^{(n)} x^{n-1-l} = \sum_{l=0}^{n-1} \lambda^l \binom{n-1}{l} B_l^{(n)} x^{n-l}, \quad (3.20)
\end{aligned}$$

where  $B_n^{(\alpha)}$  are the higher-order Bernoulli numbers defined by the generating function

$$\left( \frac{t}{e^t - 1} \right)^\alpha = \sum_{n=0}^{\infty} B_n^{(\alpha)} \frac{t^n}{n!}.$$

Thus, by (3.20), we get

$$\begin{aligned} b_{n,\lambda}(x) &= \sum_{l=0}^{n-1} \lambda^l \binom{n-1}{l} B_l^{(n)} \frac{1}{2-e^t} x^{n-l} \\ &= \sum_{l=0}^{n-1} \lambda^l \binom{n-1}{l} B_l^{(n)} b_{n-l}(x), \quad (n \geq 1). \end{aligned} \quad (3.21)$$

Therefore, by (3.21), we obtain the following theorem.

**Theorem 3.5.** *For  $n \geq 1$ , we have*

$$\begin{aligned} b_{n,\lambda}(x) &= \sum_{l=0}^{n-1} \lambda^l \binom{n-1}{l} B_l^{(n)} b_{n-l,\lambda}(x) \\ &= \sum_{l=0}^n \sum_{m=0}^{n-l} \lambda^l \binom{n-1}{l} \binom{n-l}{m} B_l^{(n)} b_m x^{n-l-m}. \end{aligned}$$

Let  $S_n(x) \sim (1, \frac{1}{\lambda}(e^{\lambda t} - 1))$ . Then, by (2.5), we get

$$\begin{aligned} \sum_{n=0}^{\infty} S_n(x) \frac{t^n}{n!} &= e^{\frac{x}{\lambda} \log(1+\lambda t)} = (1 + \lambda t)^{\frac{x}{\lambda}} \\ &= \sum_{n=0}^{\infty} \left(\frac{x}{\lambda}\right)_n \lambda^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{x}{\lambda} \left(\frac{x}{\lambda} - 1\right) \cdots \left(\frac{x}{\lambda} - n + 1\right) \lambda^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} x(x - \lambda)(x - 2\lambda) \cdots (x - (n-1)\lambda) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \end{aligned} \quad (3.22)$$

where  $(x)_{0,\lambda} = 1$ ,  $(x)_{n,\lambda} = x(x - \lambda)(x - 2\lambda) \cdots (x - (n-1)\lambda)$ ,  $(n \geq 1)$ . From (3.22), we note that  $(x)_{n,\lambda} \sim (1, \frac{1}{\lambda}(e^{\lambda t} - 1))$ . Now, we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!} &= (1 + \lambda t)^{\frac{x}{\lambda}} = e^{\frac{x}{\lambda} \log(1+\lambda t)} \\ &= \sum_{m=0}^{\infty} \left(\frac{x}{\lambda}\right)^m \frac{1}{m!} (\log(1 + \lambda t))^m \\ &= \sum_{n=0}^{\infty} \left( \sum_{m=0}^n \lambda^{n-m} S_1(n, m) x^m \right) \frac{t^n}{n!}. \end{aligned} \quad (3.23)$$

Comparing the coefficients on both sides of (3.23), we have

$$(x)_{n,\lambda} = \sum_{m=0}^n \lambda^{n-m} S_1(n, m) x^m, \quad (n \geq 0). \quad (3.24)$$

For  $(2 - e^t)b_{n,\lambda}(x) \sim (1, \frac{1}{\lambda}(e^{\lambda t} - 1))$ ,  $(x)_{n,\lambda} \sim (1, \frac{1}{\lambda}(e^{\lambda t} - 1))$ , we get

$$(2 - e^t)b_{n,\lambda}(x) = (x)_{n,\lambda} = \sum_{m=0}^n \lambda^{n-m} S_1(n, m) x^m. \quad (3.25)$$

By (3.25), we easily see that

$$\begin{aligned} b_{n,\lambda}(x) &= \sum_{m=0}^n \lambda^{n-m} S_1(n, m) \frac{1}{2 - e^t} x^m \\ &= \sum_{m=0}^n \lambda^{n-m} S_1(n, m) b_m(x), \quad (n \geq 0). \end{aligned} \quad (3.26)$$

**Corollary 3.6.** *For  $n \geq 0$ , we have*

$$b_{n,\lambda}(x) = \sum_{m=0}^n \lambda^{n-m} S_1(n, m) b_m(x).$$

By (2.9) and (3.3), we get

$$\begin{aligned} b_{n+1,\lambda}(x) &= \left( x - \frac{(-e^t)}{2 - e^t} \right) \frac{1}{e^{\lambda t}} b_{n,\lambda}(x) \\ &= \left( x + \frac{e^t}{2 - e^t} \right) b_{n,\lambda}(x - \lambda) = x b_{n,\lambda}(x - \lambda) + \frac{e^t}{2 - e^t} b_{n,\lambda}(x - \lambda) \\ &= x b_{n,\lambda}(x - \lambda) + \frac{e^t - 2 + 2}{2 - e^t} b_{n,\lambda}(x - \lambda) \\ &= x b_{n,\lambda}(x - \lambda) - b_{n,\lambda}(x - \lambda) + \frac{2}{2 - e^t} b_{n,\lambda}(x - \lambda). \end{aligned} \quad (3.27)$$

From (3.10), we note that

$$\begin{aligned}
\frac{1}{2-e^t} b_{n,\lambda}(x-\lambda) &= \frac{1}{2-e^t} \sum_{k=0}^n \sum_{m=0}^k \binom{n}{k} \lambda^{k-m} S_1(k, m) b_{n-k,\lambda}(x-\lambda)^m \\
&= \frac{1}{2-e^t} \sum_{k=0}^n \sum_{m=0}^k \binom{n}{k} \lambda^{k-m} S_1(k, m) b_{n-k,\lambda} \sum_{l=0}^m \binom{m}{l} (-1)^l \lambda^l x^{m-l} \\
&= \sum_{k=0}^n \sum_{m=0}^k \sum_{l=0}^m \binom{n}{k} \binom{m}{l} \lambda^{k-m+l} S_1(k, m) b_{n-k,\lambda} (-1)^l \frac{1}{2-e^t} x^{m-l} \\
&= \sum_{k=0}^n \sum_{m=0}^k \sum_{l=0}^m \binom{n}{k} \binom{m}{l} \lambda^{k-m+l} S_1(k, m) b_{n-k,\lambda} (-1)^l b_{m-l}(x).
\end{aligned} \tag{3.28}$$

By (3.27) and (3.28), we get

$$\begin{aligned}
b_{n+1,\lambda}(x) &= x b_{n,\lambda}(x-\lambda) - b_{n,\lambda}(x-\lambda) + \frac{2}{2-e^t} b_{n,\lambda}(x-\lambda) \\
&= x b_{n,\lambda}(x-\lambda) - b_{n,\lambda}(x-\lambda) \\
&\quad + 2 \sum_{k=0}^n \sum_{m=0}^k \sum_{l=0}^m \binom{n}{k} \binom{m}{l} \lambda^{k-m+l} S_1(k, m) b_{n-k,\lambda} (-1)^l b_{m-l}(x).
\end{aligned} \tag{3.29}$$

Therefore, by (3.29), we obtain the following theorem.

**Theorem 3.7.** *For  $n \geq 0$ , we have*

$$\begin{aligned}
&b_{n+1,\lambda}(x) - x b_{n,\lambda}(x-\lambda) + b_{n,\lambda}(x-\lambda) \\
&= 2 \sum_{k=0}^n \sum_{m=0}^k \sum_{l=0}^m \binom{n}{k} \binom{m}{l} \lambda^{k-m+l} S_1(k, m) b_{n-k,\lambda} (-1)^l b_{m-l}(x).
\end{aligned}$$

**Remark.** From  $b_{n,\lambda}(x) \sim (2-e^t, \frac{1}{\lambda}(e^{\lambda t}-1))$  and (2.14), we note that

$$\begin{aligned}
\frac{d}{dx} b_{n,\lambda}(x) &= \sum_{l=0}^{n-1} \binom{n}{l} < \frac{1}{\lambda} \log(1+\lambda t) |x^{n-l} > b_{l,\lambda}(x) \\
&= \sum_{l=0}^{n-1} \binom{n}{l} < \sum_{m=1}^{\infty} \frac{(-\lambda)^{m-1} t^m}{m} |x^{n-l} > b_{l,\lambda}(x) \\
&= \sum_{l=0}^{n-1} \binom{n}{l} (-\lambda)^{n-l-1} (n-l-1)! b_{l,\lambda}(x), \quad (n \geq 1).
\end{aligned} \tag{3.30}$$

Let

$$\mathbb{P}_n = \{p(x) \in \mathbb{C}[x] \mid \deg p(x) \leq n\}, \quad (n \geq 0). \tag{3.31}$$

For  $p(x) \in \mathbb{P}_n$ , let

$$p(x) = \sum_{l=0}^n a_l b_{l,\lambda}(x). \quad (3.32)$$

From (3.3), we note that

$$\langle (2 - e^t)(\frac{1}{\lambda}(e^{\lambda t} - 1))^m | b_{n,\lambda}(x) \rangle = n! \delta_{n,m}, \quad (n, m \geq 0). \quad (3.33)$$

By (3.32) and (3.33), we get, for  $0 \leq m \leq n$ ,

$$\begin{aligned} & \langle (2 - e^t)(\frac{1}{\lambda}(e^{\lambda t} - 1))^m | p(x) \rangle \\ &= \sum_{l=0}^n a_l \langle (2 - e^t)(\frac{1}{\lambda}(e^{\lambda t} - 1))^m | b_{l,\lambda}(x) \rangle \\ &= \sum_{l=0}^n a_l l! \delta_{l,m} = a_m m!. \end{aligned} \quad (3.34)$$

From (3.34), we have

$$a_m = \frac{1}{m!} \langle (2 - e^t)(\frac{1}{\lambda}(e^{\lambda t} - 1))^m | p(x) \rangle, \quad (3.35)$$

where  $n \geq m \geq 0$ . Therefore, we obtain the following theorem.

**Theorem 3.8.** *For  $p(x) \in \mathbb{P}_n$ , we have*

$$p(x) = \sum_{m=0}^n a_m b_{m,\lambda}(x), \quad (n \geq 0),$$

where

$$a_m = \frac{1}{m!} \langle (2 - e^t)(\frac{1}{\lambda}(e^{\lambda t} - 1))^m | p(x) \rangle.$$

Let us take  $p(x) = B_n(x) \in \mathbb{P}_n$ , ( $n \geq 0$ ). Then, by Theorem 8, we get

$$B_n(x) = \sum_{m=0}^n a_m b_{m,\lambda}(x), \quad (n \geq 0), \quad (3.36)$$

where

$$\begin{aligned}
a_m &= \frac{1}{m!} \langle (2 - e^t) \left( \frac{1}{\lambda} (e^{\lambda t} - 1) \right)^m | B_n(x) \rangle \\
&= \frac{1}{m!} \langle (2 - e^t) \lambda^{-m} m! \sum_{k=m}^{\infty} S_2(k, m) \lambda^k \frac{t^k}{k!} | B_n(x) \rangle \\
&= \sum_{k=m}^n \lambda^{k-m} S_2(k, m) \frac{1}{k!} \langle 2 - e^t | t^k B_n(x) \rangle \\
&= \sum_{k=m}^n \lambda^{k-m} S_2(k, m) \binom{n}{k} \langle 2 - e^t | B_{n-k}(x) \rangle \\
&= \sum_{k=m}^n \lambda^{k-m} S_2(k, m) \binom{n}{k} (2B_{n-k} - B_{n-k}(1)) \\
&= \sum_{k=m}^n \lambda^{k-m} S_2(k, m) \binom{n}{k} (B_{n-k} - \delta_{1, n-k}) \\
&= \sum_{k=m}^n \lambda^{k-m} S_2(k, m) \binom{n}{k} B_{n-k} - n \lambda^{n-m-1} S_2(n-1, m).
\end{aligned} \tag{3.37}$$

Therefore, by (3.36) and (3.37), we obtain the following theorem.

**Theorem 3.9.** *For  $n \geq 0$ , we have*

$$\begin{aligned}
B_n(x) &= \sum_{m=0}^n \sum_{k=m}^n \lambda^{k-m} S_2(k, m) \binom{n}{k} B_{n-k} b_{m, \lambda}(x) \\
&\quad - n \sum_{m=0}^{n-1} \lambda^{n-m-1} S_2(n-1, m) b_{m, \lambda}(x).
\end{aligned}$$

For  $b_{n, \lambda}(x) \sim (2 - e^t, \frac{1}{\lambda}(e^{\lambda t} - 1))$ ,  $(x)_{n, \lambda} \sim (1, \frac{1}{\lambda}(e^{\lambda t} - 1))$ , ( $n \geq 0$ ), we have

$$(x)_{n, \lambda} = \sum_{m=0}^n C_{n, m} b_{m, \lambda}(x), \quad (n \geq 0), \tag{3.38}$$



where

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!} \langle (2 - e^{\frac{1}{\lambda} \log(1+\lambda t)})^{-1} (\frac{1}{\lambda} (e^{\log(1+\lambda t)} - 1))^m | x^n \rangle \\
&= \frac{1}{m!} \langle (2 - (1 + \lambda t)^{\frac{1}{\lambda}})^{-1} t^m | x^n \rangle \\
&= \binom{n}{m} \langle \sum_{k=0}^{\infty} b_{k,\lambda} \frac{t^k}{k!} | x^{n-m} \rangle \\
&= \binom{n}{m} b_{n-m,\lambda}.
\end{aligned} \tag{3.39}$$

Therefore, by (3.39), we obtain the following theorem.

**Theorem 3.10.** *For  $n \geq 0$ , we have*

$$(x)_{n,\lambda} = \sum_{m=0}^n \binom{n}{m} b_{n-m,\lambda} b_{m,\lambda}(x).$$

The Korobov polynomials  $K_{n,\lambda}(x)$  for  $\lambda \neq 0, 1$  are also defined in terms of generating function by

$$\frac{\lambda t}{(1+t)^\lambda - 1} (1+t)^x = \sum_{n=0}^{\infty} K_{n,\lambda}(x) \frac{t^n}{n!}, \quad (\text{see [6]}). \tag{3.40}$$

Thus, by (3.40), we get

$$K_{n,\lambda}(x) \sim \left( \frac{e^{\lambda t} - 1}{\lambda(e^t - 1)}, e^t - 1 \right), \tag{3.41}$$

For  $K_{n,\lambda}(x) \sim \left( \frac{e^{\lambda t} - 1}{\lambda(e^t - 1)}, e^t - 1 \right)$ ,  $b_{n,\lambda}(x) \sim (2 - e^t, \frac{1}{\lambda}(e^{\lambda t} - 1))$ , we have

$$K_{n,\lambda}(x) = \sum_{m=0}^n C_{n,m} b_{m,\lambda}(x), \quad (n \geq 0), \tag{3.42}$$

where

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!} \left\langle (1-t) \frac{\lambda t}{(1+t)^\lambda - 1} \left( \frac{1}{\lambda} ((1+t)^\lambda - 1) \right)^m \middle| x^n \right\rangle \\
&= \frac{\lambda^{-m}}{m!} \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \left\langle (1-t) \frac{\lambda t}{(1+t)^\lambda - 1} (1+t)^{l\lambda} \middle| x^n \right\rangle \\
&= \frac{\lambda^{-m}}{m!} \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \left\langle (1-t) \left| \sum_{j=0}^n K_{j,\lambda}(l\lambda) \frac{t^j}{j!} x^n \right. \right\rangle \\
&= \frac{\lambda^{-m}}{m!} \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \sum_{j=0}^n K_{j,\lambda}(l\lambda) \binom{n}{j} (\delta_{j,n} - \delta_{j,n-1}) \\
&= \frac{\lambda^{-m}}{m!} \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} (K_{n,\lambda}(l\lambda) - nK_{n-1,\lambda}(l\lambda)).
\end{aligned} \tag{3.43}$$

Therefore, by (3.42) and (3.43), we obtain the following theorem.

**Theorem 3.11.** *For  $n \geq 0$ , we have*

$$K_{n,\lambda}(x) = \sum_{m=0}^n \left( \frac{\lambda^{-m}}{m!} \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} (K_{n,\lambda}(l\lambda) - nK_{n-1,\lambda}(l\lambda)) \right) b_{m,\lambda}(x).$$

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DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE TIANJIN POLYTECHNIC UNIVERSITY,  
TIANJIN 300160, CHINA, DEPARTMENT OF MATHEMATICS, KWANGWOON UNIVERSITY, SEOUL  
139-701, REPUBLIC OF KOREA

*E-mail address:* `tkkim@kw.ac.kr`

DEPARTMENT OF MATHEMATICS, SOGANG UNIVERSITY, SEOUL 121-742, REPUBLIC OF KO-  
REA

*E-mail address:* `dskim@sogang.ac.kr`